From Static Locus Problems to Exploring Mathematics with Technological Tools

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Abstract

This is an expanded version of the paper [11]. We discuss two problems taken from practice problems preparation guides for entrance examinations into Chinese universities *[8]. We see how original problems in 2D, stated in an exam static and uninspired settings, can be extended to other interesting cases in 2D and more challenging corresponding problems in 3D for students to explore with the help of a Dynamic Geometry Software (DGS) and a Computer Algebra System (CAS). We use a DGS to construct the locus or locus surface geometrically, and use a CAS to verify our locus or locus surface analytically. We shall see that with the innovative use of technological tools, mathematics can be made more fun, accessible, challenging and applicable to a broader group of students and teachers alike. Finally, we attempt to make these problems relevant to real-life applications, and invite readers to imagine some other interpretations for the proposed situations. A video clip summarizing those examples using GInMA [8] can be found in [S13].*

1 Introduction

Ever since the document of *Innovation on Mathematics Curriculum and Textbooks in China* was released in 2006 (see *[1]*), technological tools have been adopted for explorations in many high schools in China. However, because college entrance examination still play a crucial component for students' future success, students and parents wonder how activities involving exploration could help students improve their exam grades. They are concerned with the fact that the nature of the assessment methods that students face in many countries does not reward exploration. However, we cannot ignore the fact that innovation and understanding do not always come from drills or rote-type learning, but from exploration. The author believes that we should recognize the importance of stimulating the discussion of mathematics and its applications through timely use of technological tools (see *[7]* or *[10])*. In this paper, we present two practice problems found from the preparation guides for entrance examinations into

Chinese universities [8]. To make these problems more accessible, interesting and challenging at the same time, we propose the use of a DGS in order to construct geometrically the potential curve for a locus problem. While students may be able to solve locus equations by hand when the problems are simple, they promptly can discover that finding the algebraic equation for a locus by hand is virtually impossible when problems become more complicated. Consequently, they can appreciate the need of a DGS for construction purposes and a CAS to validate whether the algebraic equation for the locus matches with the plot that was obtained from the DGS. We shall see that the problems discussed in 2D can be extended to respective 3D scenarios when students have some knowledge of multivariable calculus. Furthermore, the locus problems can be linked to real-life scenarios and the author manages some possibilities and invites readers to develop more applications on their own.

2 First Problem and Some Extended Activities

The original statement of this first problem is stated as follows: *Given a unit circle centered at* $(0, 0)$ *and a fixed point at* $A = (2, 0)$. *Let* Q *be a moving point on the unit circle* C. *Find the locus* M *which is the intersection between the angle bisector* QOA *and line segment* QA. It is an easy exercise to verify that the locus of point M in this problem is a circle, which we leave as an exercise for the readers. Moreover, it is natural to imagine when DGS and CAS tools are available for students in a classroom as a project to explore, they may quickly pose 'what if' scenarios. For example, we consider the following case:

Example 1 *Given an ellipse* C*:* $[x(t), y(t)] = [a\cos(t), b\sin(t)]$ *and a fixed point* $A = (p, q)$. *Let* Q *be a moving point on the ellipse. Find the locus of the point* M *which is the intersection between the bisector* QOA *and line segment* QA.

Students may use their favorite DGS to construct a trace of the locus M without too much trouble. We use GInMA (*[4]*) to illustrate one of the infinite possibilities for the locus, which is shown in red color in Figure 1 below.

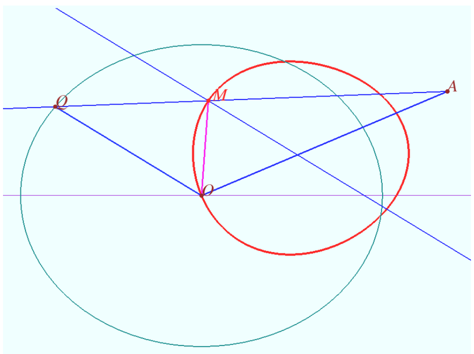

Figure 1. Locus, bisection and an ellipse.

We note that a DGS allows users to drag the moving point Q and see how the corresponding locus M moves accordingly. Similarly, we can also make the point A movable and see how the locus changes accordingly (see Figures 2(a) and 2(b)). Being able to visualize and manipulate a dynamic graph constructed from a DGS will allow students to quickly comprehend the original question and make additional observations based on *what if* scenarios. The next step students can do is to see is to see if they can apply their mathematical knowledge to derive the equation for the locus analytically, and verify if what they saw earlier from DGS is reasonable. We shall see below that the locus can be found with a little help from geometry and familiarity with parametric equations. First, we construct a line passing through M that is parallel to OQ, and label the intersection between this line and OA as B. It follows from the Angle Bisector Theorem that $\frac{MA}{MQ} = \frac{OA}{OQ}$. Suppose we denote $\frac{OA}{OQ}$ by $k(t)$, which is not constant but a function $k(t)$ in this case. We see that OBM is an isosceles triangle with $MB = OB$. Therefore, we have

$$\frac{MA}{MQ} = \frac{AB}{OB} = \frac{AB}{MB} = \frac{OA}{OQ} = k(t).$$

Since $\overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{BA} = \overrightarrow{OB} + k(t)\overrightarrow{OB}$, we see

$$\begin{aligned}
\overrightarrow{OB} &= \frac{1}{k(t)+1}\overrightarrow{OA} \\
\overrightarrow{BA} &= \overrightarrow{OA} - \overrightarrow{OB} = \overrightarrow{OA} - \frac{1}{k(t)+1}\overrightarrow{OA} \\
&= \frac{k(t)}{k(t)+1}\overrightarrow{OA}.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\overrightarrow{BM} &= \frac{k(t)}{k(t)+1}\overrightarrow{OQ} \text{ and} \\
\overrightarrow{OM} &= \overrightarrow{OB} + \overrightarrow{BM} = \frac{1}{k(t)+1}\overrightarrow{OA} + \frac{k(t)}{k(t)+1}\overrightarrow{OQ} \\
&= \frac{1}{\frac{OA}{OQ}+1}\overrightarrow{OA} + \frac{\frac{OA}{OQ}}{\frac{OA}{OQ}+1}\overrightarrow{OQ} \\
&= \frac{OQ}{OA+OQ}\overrightarrow{OA} + \frac{OA}{OA+OQ}\overrightarrow{OQ}. \qquad (1)
\end{aligned}$$

In addition, because $OQ = \sqrt{a^2\cos^2 t + b^2\sin^2 t}$ and $OA = \sqrt{p^2+q^2}$, the parametric equation for the locus M can be seen directly from Eq. (1) above. With the help of a CAS such as MAPLE (*[6]*) used here, we plot the locus M together with the original ellipse when

$a = 2, b = 1, p = 2$ and $q = 0$ as displayed in Figure 2(c):

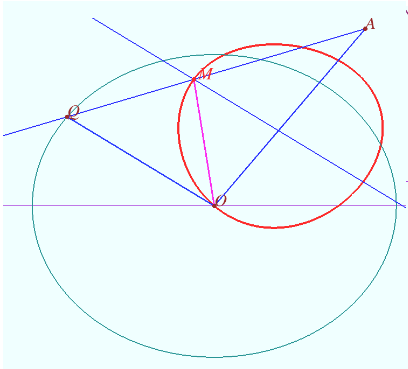

Figure 2(a). Point A is outside the ellipse.

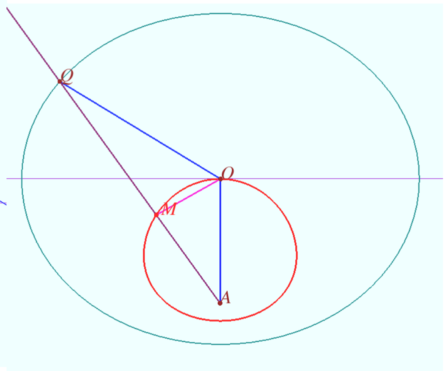

Figure 2(b). Point A is inside the ellipse.

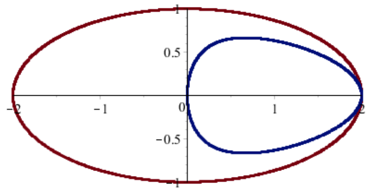

Figure 2(c). Plot generated by MAPLE.

We can pose another scenario when the ellipse is replaced by a cardioid as follows, which is left as an exercise for the reader.

Exercise 2 *Given a cardioid C of the form $[x(t), y(t)] = [(1-\cos(t))\cos(t)+1, (1-\cos(t))\sin(t)]$ and a fixed point $A = (p, q)$. Let Q be a moving point on the cardioid. Find the locus of the point M, which is the intersection between the bisector QOA and the line segment QA, when Q moves along the cardioid C.*

As we have mentioned earlier, we encourage students to use a DGS to explore their possible locus before validating analytically with their favorite CAS to see if the locus seen from a DGS matches that of the CAS. We used GInMA *[4]* to draw the locus when we varied the point A, as displayed in Figures 3(a) and 3(b) below. We also used MAPLE *[6]* to plot the locus analytically when $p = 3$ and $q = 2$ as shown in Figure 3(c).

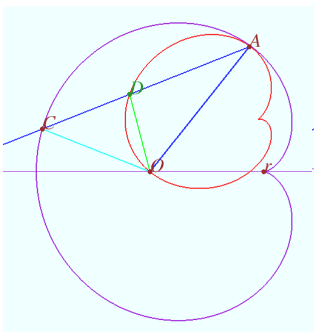

Figure 3(a). Locus and a cardioid.

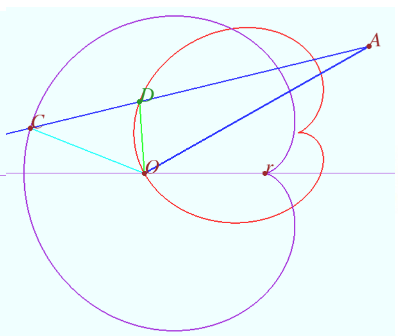

Figure 3(b). Point A is outside the cardioid.

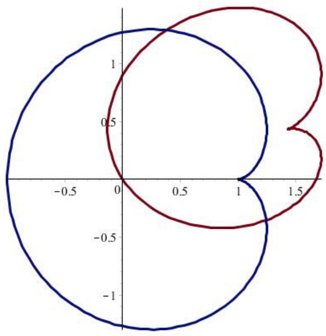

Figure 3(c). Locus generated by MAPLE.

We note that our principal tool for deriving the analytical formulae in Example 1 was the Angel Bisector Theorem and we can apply it in this setting. In particular, we get an analogous

Eq. (1) or

$$\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} = \frac{OQ}{OA+OQ}\begin{bmatrix} p \\ q \end{bmatrix} + \frac{OA}{OA+OQ}\begin{bmatrix} x_0(t) \\ y_0(t) \end{bmatrix},$$

which is a key for deriving the corresponding locus of $[x_1(t), y_1(t)]$ for a given curve $[x_0(t), y_0(t]$ and a fixed point $A = (p, q)$. Now for general $n = 0, 1, ...,$ we consider

$$\begin{bmatrix} x_{n+1}(t) \\ y_{n+1}(t) \end{bmatrix} = \frac{OQ_n}{OA+OQ_n}\begin{bmatrix} p \\ q \end{bmatrix} + \frac{OA}{OA+OQ_n}\begin{bmatrix} x_n(t) \\ y_n(t) \end{bmatrix}, \tag{2}$$

with $OQ_n = \sqrt{x_n(t)^2 + y_n(t)^2}$ and $OA = \sqrt{p^2+q^2}$. The point Q_n is seen to be a moving point on $[x_n(t), y_n(t)]$. Then the locus M_{n+1}, which is the intersection between the bisector Q_nOA and line segment Q_nA, is $\begin{bmatrix} x_{n+1}(t) \\ y_{n+1}(t) \end{bmatrix}$. For example, if $[x_0(t), y_0(t)] = [(1-\cos(t))\cos(t)+1, (1-\cos(t))\sin(t)]$ and a fixed point $A = (3, 2)$, then using MAPLE *[6]* and the code given in [S2], we derive the following interesting plots for $[x_i(t), y_i(t)]$,for $i = 0, 1, 2$ and 3, which can be seen in Figure 4 below with the help of [S2].

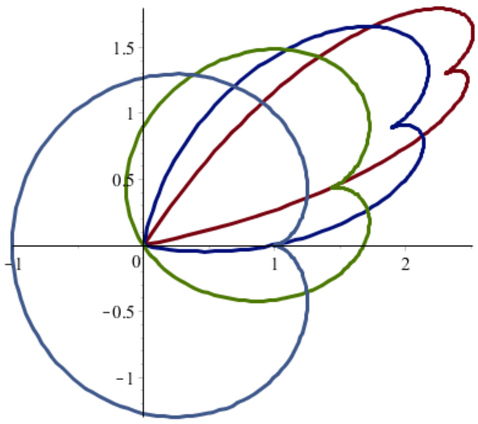

Figure 4. A sequence of plots of bisections.

2.1 Possible Real Life Interpretations in 2D

1. Consider Figures 2(a) or 3(b) and suppose an allied aircraft Q is moving along the shape of a given curve C, which could be an ellipse or a cardioid. Assume the allied aircraft carrier is set up at the point A (outside curve C), which communicates with a command center at $O = (0, 0)$. If an enemy aircraft decides to move along (roughly) the intersection between the angle bisector QOA and QA to avoid being targeted, find the possible route for the enemy.

2. A game is described as follows (see Figure 5). A light source Q, pointing at a point O, is moving along a curve C (which could be an ellipse or a cardioid as described in Example 1 or Exercise 2 respectively). The reflected light ray always points in the direction of $\overrightarrow{OA}$,

where A is a fixed point. It is known that a target M is always staying at the intersection of the line segment of AQ and the normals of "mirror sticks". (These mirror sticks are shown in line segments L or $L'$when point Q is moved to Q' in Figure 3(d)). The game is for you to maneuver the mirror sticks so you can hit the target M.

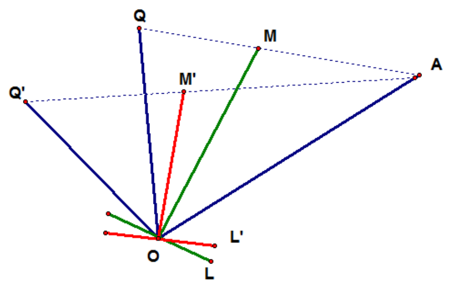

Figure 5. A game and light reflections.

3. Use your imagination to interpret your real-life scenarios.

2.2 Extensions to 3D scenarios

Here we describe a problem for students to explore once they have knowledge of parametric equations for surfaces. We shall see that the bisection theorem used in 2D is still valid in 3D explorations. Specifically, we explore scenarios when we replace the ellipse and cardioid by an ellipsoid and a cardioid surface respectively. We state these two scenarios as follows:

Example 3 *Given an ellipsoid S: $x(\theta,\varphi) = a\sin\varphi\cos\theta, y(\theta,\varphi) = b\sin\varphi\sin\theta$ and $z(\theta,\varphi) = c\cos\varphi$, and a fixed point $A = (p,q,r)$. We pick a moving point Q on the ellipsoid S. Find the locus M which is the intersection between the bisector QOA and line segment QA.*

We show, with the aid of GInMA *[4]*, a scenario for the locus of M in Figure 6(a). We note that Eq.(8) can be extended to find the parametric equation for the locus surface in 3D as follows:

$$\begin{bmatrix} X(\theta,\varphi) \\ Y(\theta,\varphi) \\ Z(\theta,\varphi) \end{bmatrix} = \frac{OQ}{OA+OQ}\begin{bmatrix} p \\ q \\ r \end{bmatrix} + \frac{OA}{OA+OQ}\begin{bmatrix} x(\theta,\varphi) \\ y(\theta,\varphi) \\ z(\theta,\varphi) \end{bmatrix}$$

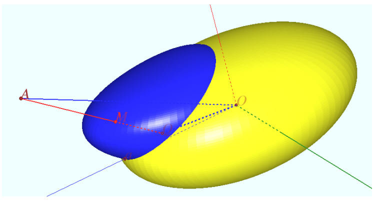

Figure 6(a). Locus surface and an ellipsoid.

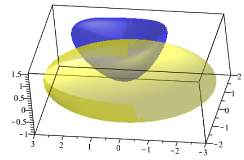

Figure 6(b). A non-convex locus generated by Maple.

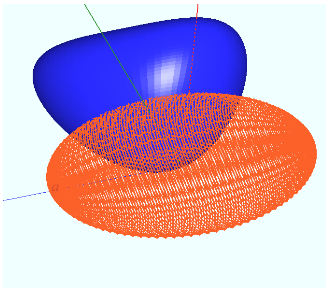

Figure 6(c). A non-convex locus generated by GInMA.

Through various exploration by adjusting the shape of the ellipsoid (a, b, c) and the fixed point $A = (p, q, r)$, we found an interesting non-convex locus when $(a, b, c) = (3, 2, 1)$ and $(p, q, r) = (3, 1, 1)$. Both, the ellipsoid and the locus are shown using MAPLE *[6]* and GInMA *[4]* in Figures 6(b) and 6(c) respectively. We pose another scenario when we replace the ellipsoid by a cardioid surface, which we leave as an exercise as follows:

Exercise 4 *We construct a cardioid surface S by rotating the 2D parametric curve of $[x(t), y(t)]$, where $x(t) = a(1-\cos t)\cos(t)+a$, $y(t) = a(1-\cos t)\sin(t)$ and $t \in [0, 2\pi]$, around the $x-axis$. We let A be a fixed point and pick a moving point Q on the cardioid surface S. Use a DGS or CAS to find the locus M which is the intersection between the angle bisector QOA and the line segment QA.[Hint: We see that the cross sections of the surface S are circles parallel to the $yz-plane$, whose centers are on the $x-axis$ with radius $y(t)$. Let angle φ be the angle between the vector from center to the point on each cross section and the positive $y-axis$, then the parametric surface becomes $[x(t), y(t)\cos\varphi, y(t)\sin\varphi]$, where $t \in [0, 2\pi]$ and $\varphi \in [0, \pi]$.*

Once again, we use MAPLE *[6]* to sketch the locus surface (in blue) and the original cardioid surface, with $A = (1, 2, 3)$ and $a = 1$, as seen in Figure 7.

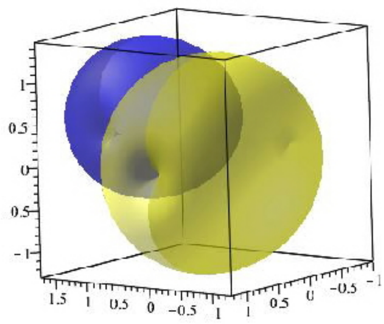

Figure 7. Locus surface generated by MAPLE.

2.3 Possible Real Life Applications in 3D

1. An allied aircraft Q is moving along the shape of a given ellipsoid or cardioid surface. The allied aircraft carrier is set up at the point A, which communicates with a command center O at the center of the ellipsoid or cardioid surface. An enemy aircraft decides to move along at the intersection between the angle bisector QOA and QA to avoid being hit. Find the possible route for the enemy.

2. A game is described as follows: A light source Q, pointing at a point $O = (0, 0, 0)$, is moving along a surface S (it could be an ellipsoid or a cardioid surface as described in Example 3 or Exercise 4). The reflected light ray is always kept in the direction of $\overrightarrow{OA}$, where A is a fixed point in space. It is known that the target M is always staying at the intersection between the line segment of AQ and the normals of mirror planes when Q is moving along S. The game is for you to maneuver the mirror planes so that the target M can be hit.

3. Use your imagination to interpret your real-life scenarios.

3 Second Problem and Some Extended Activities

In this section we enunciate our second problem and with the help of a DGS and a CAS, derive from it several different and more general scenarios in 2D as well as in 3D settings. We also recall that the original problem was taken from a practice-problems guide for Chinese universities entrance examinations. The original statement for this problem is stated as follows:

Example 5 *We are given two concentric circles centered at $O = (0, 0)$ with radii of 1 and 2 respectively. We are given a moving point A on the unit circle. We extend the line OA to intersect at a point B on the outer circle. We then construct the line l_1 passing through B and parallel to the $y-axis$. Finally, we construct the line l_2 passing through the point A and parallel to the $x-axis$. Find the locus for the point P that is the intersection between l_1 and l_2.*

Note that it is quite easy to solve this problem with some help of a DGS and students can use their favorite one in order to working it out. In fact, they will quickly realize that this problem can serve as one way of constructing an ellipse from two concentric circles as shown in

Figure 8.

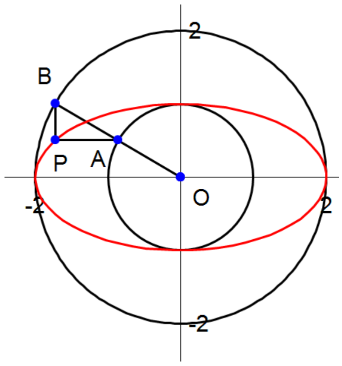

Figure 8. Generating an ellipse from two concentric circles.

Although finding the locus is quite an elementary exercise that requires only a simple knowledge of trigonometry, this problem actually serves a good purpose of understanding how the parametric equation for an ellipse can be derived. Assume the radii of inner and outer circles to be b and a respectively, then it is quite simple to recognize (see *[2]* and Figures 9(a) and 9(b)) that the locus of the desired ellipse will be of the form of $[a\cos t, b\sin t]$.It is interesting to note from *[2] that 'For extreme accuracy it's probably the best method. It's convenient for use on a drafting board with T-square and triangles'.*

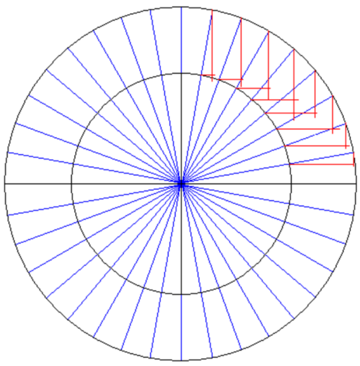

Figure 9(a). Construction of an ellipse.

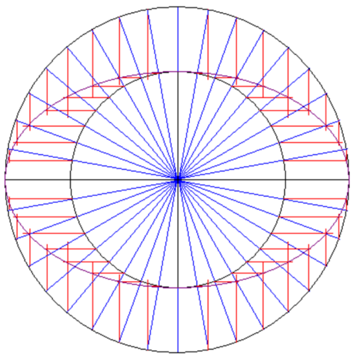

Figure 9(b). Locus derived from the construction.

3.1 Extended 2D Scenarios

Now, it is natural to now ask what the locus would be if we replace the outer circle by an ellipse. Here we propose some more general settings than the original problem and again, we use the DGS GInMA [4] and Geometry Expression [2], as well as the CAS MAPLE [6] for our constructions.

Example 6 *We are given a circle C with radius r_0 and centered at $O = (0,0)$, and an ellipse* *of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, that is outside the given circle. Let A be a moving point on the circle and construct the line OA to intersect at a point B on the ellipse. We construct the line l_1 passing through B and parallel to the $y-axis$. Next we construct the line l_2 passing through the point A and parallel to the $x-axis$. (a) Find the locus for the point P that is the intersection of the lines l_1 and l_2. (b) Find the point B which yields the maximum area for the triangle APB.*

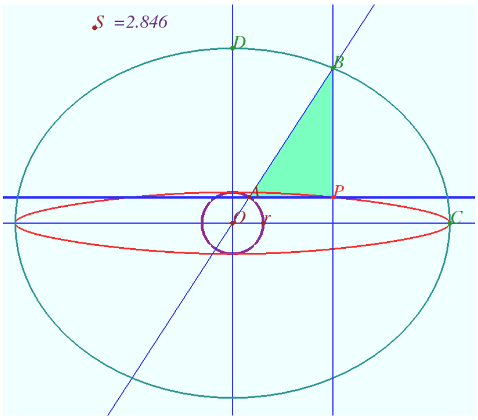

Figure 10. Generating the locus from a circle and an ellipse.

We note that part (a) of this problem can be solved by hand without too much work. We write $A=(A_x,A_y), B=(B_x,B_y)$, and let $OB=r, \measuredangle BOC=\theta$. Then $B=(a\cos\theta, b\sin\theta)$. It is easy to see that $OB^2=a^2\cos^2\theta+b^2\sin^2\theta=a^2\cos^2\theta+b^2(1-\cos^2\theta)=b^2+(a^2-b^2)\cos^2\theta$. Thus $r^2=b^2+(a^2-b^2)\frac{1+\cos 2\theta}{2}$, which leads to

$$OB=r=\frac{\sqrt{2}ab}{\sqrt{a^2+b^2-(a^2-b^2)\cos 2\theta}}. \tag{3}$$

If we write down the locus for the point $P=(P_x,P_y)$, then $(P_x)^2=\left(\frac{\sqrt{2}ab}{\sqrt{a^2+b^2-(a^2-b^2)\cos 2\theta}}\right)^2\cos^2\theta=\dfrac{2a^2b^2\cos^2\theta}{a^2+b^2-(a^2-b^2)\cos 2\theta}$ and $(P_y)^2=r_0^2\sin^2\theta$. For part (b), the area of ABP is the absolute value of

$$\frac{1}{2}(AP)(BP)=\frac{1}{2}(P_x-A_x)(B_y-P_y)=\frac{1}{2}(r\cos\theta-r_0\cos\theta)(r\sin\theta-\sqrt{r_0}\sin\theta). \tag{4}$$

Now we substitute r in Eq. (3) into the area of ABP and use a CAS to simplify Eq. (4) to the following form:

$$\frac{1}{4}\sin 2\theta\left(\frac{\sqrt{2}ab}{\sqrt{a^2+b^2-(a^2-b^2)\cos 2\theta}}-r_0\right)^2.$$

The locus corresponds to an ellipse is sketched in Figure 10 with the aid of GInMA *[4]*. If we use a CAS such as *[6]* with specific numeric values of $a=5, b=4$ and $r_0=\frac{1}{\sqrt{2}}$, we find the maximum area of ABP to be 3.5631, which occurs when θ is about 0.655308 radians or 37.5464 degrees. In the following example, we investigate a similar locus problem but the respective centers for the two curves are at different locations, which we stated the problem as follows:

Example 7 *We are given a circle C^* centered at $O = (0,0)$ with radius r_0, and a cardioid which resembles the shape of $r = a(1-\cos\theta)$, where $\theta \in [0, 2\pi]$ enclosing the given circle C^* as shown in Figure 11(a). We are given a moving point A on the circle. Suppose we construct the line OA to intersect at a point B on the cardioid. We construct the line l_1 passing through B and is parallel to $y-axis$. Next we construct the line l_2 passing through the point A and is parallel to $x-axis$. (a) Find the locus for the point P that is the intersection of the lines l_1 and l_2. (b) Find the point B on the cardioid which yields the maximum area for the triangle APB.*

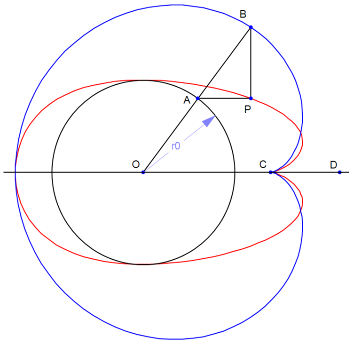

Figure 11(a). Locus generated with Geometry Expression [3].

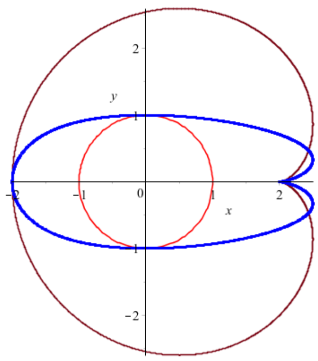

Figure 11(b). Locus generated by Maple [6].

First, we notice in Figure 11(a), the cardioid enclosing the circle, is given in parametric form of $r = a(1-\cos\theta)$; it is centered at the point C and the circle C^* is centered at $(0,0)$. If we use $O = (0,0)$ as the center of the cardioid enclosing the circle, we may write the parametric equation $[x(\theta), y(\theta)]$ for such cardioid as $x(\theta) = a(1-\cos\theta)\cos(\theta) + OC$ and $y(\theta) = a(1-\cos\theta)\sin(\theta)$ with $OC > r_0$. Now, we let $\theta = \measuredangle BOC, \varphi = \measuredangle BCD$, $OB = R, OC = a$. We will next express R in terms of a and angle φ. We write locus $P = (P_x, P_y)$ for the locus and set the points $A = (A_x, A_y)$ and $B = (B_x, B_y)$. It is clear that $P_y = A_y = r_0 \sin\theta$ and $P_x = B_x$. Also notice that the original cardioid, in blue, in Figure 11(a) can be represented by $r = a(1-\cos\varphi)$. We observe that $B_x = R\cos\theta = a + r\cos\varphi$ and $B_y = R\sin\theta = r\sin\varphi$, which leads to

$$\begin{aligned} R^2 &= a^2 + 2ar\cos\varphi + r^2 \\ &= a^2 + 2a\left(a(1-\cos\varphi)\right)\cos\varphi + a^2(1-\cos\varphi)^2 \\ &= a^2\left(2-\cos^2\varphi\right). \end{aligned}$$

This implies

$$R = a\sqrt{2-\cos^2\varphi}.$$

On the other hand, since $P_x = B_x$, we see

$$\begin{aligned} \frac{P_x}{a} &= 1 + \frac{r}{a}\cos\varphi = 1 + (1-\cos\varphi)\cos\varphi = \sin^2\varphi + \cos\varphi, \\ P_x &= a\left(\sin^2\varphi + \cos\varphi\right). \end{aligned} \tag{5}$$

Furthermore, we have

$$\begin{aligned}
\frac{P_y}{r_0} &= \sin\theta = \frac{r}{R}\sin\varphi = \frac{r\sin\varphi}{a\sqrt{2-\cos^2\varphi}} = \frac{\sin\varphi\,(1-\cos\varphi)}{\sqrt{2-\cos^2\varphi}} \text{ and} \\
P_y &= r_0\left(\frac{\sin\varphi\,(1-\cos\varphi)}{\sqrt{2-\cos^2\varphi}}\right). \qquad (6)
\end{aligned}$$

Notice that Eqs (5) and (6) give representation for the locus P in terms of angle φ. We plot the locus $[P_x, P_y]$ together with cardioid and circle when $r_0 = 1$ and $OC = 2$ in Figure 11(b) with the help of MAPLE *[6]*. If we make the substitution of $t = \tan\frac{\varphi}{2}$, then we can see that $t^4 + 4t^3\cot\theta - 4t^2 - 1 = 0$, which yields

$$\frac{P_x}{a} = \frac{2t^2}{1+t^2} \text{ and } \frac{P_y}{a} = \sin\theta.$$

The Eqs. (5) and (6) represent the locus P in terms of angle φ. The sketch of the locus corresponding to a cardioid is shown using Geometry Expression *[3]* in Figure 11(a). We leave it as an exercise for the reader to find the maximum area for the triangle ABP.

3.2 Possible Real Life Interpretation in 2D

1. A sea rock is similar to the shape of half of a circle. An airplane is flying on the path of C (either a bigger circle, an ellipse or a cardioid that is enclosing the circle). The airplane decides to lower a basket tied to a vertical ladder intending to rescue people standing at a point A on the sea rock. But because of the tides, the sea rock may be covered by various levels of water at times. The tides are assumed to be lines parallel to the sea level. We assume those people who need to be rescued from the sea rock may need to swim to the location where the basket is lowered. (a) Find the locus of the rescuing basket. (b) Furthermore, If it is decided that the best place the airplane should lower the basket is at the point where the area of the triangle ABP reaches its maximum, find the place where the airplane should lower the basket.

2. Exercise for the reader: Use your imagination to interpret one of your real-life scenarios.

3.3 Extensions to 3D scenarios

In view of three cases we just described in 2D, we now naturally extend these scenarios to 3D. Specifically, we state these 3D scenarios as follows:

Example 8 *We are given two concentric spheres centered at $O = (0,0,0)$ of radii of a and b (with $a < b$) respectively. See Figure 12, that is generated by GInMA [4] below. The unit sphere is depicted in blue and the sphere of radius 2 is the one in yellow. We are given a moving point A on the unit sphere and extend the ray OA to intersect the outer sphere at a point B. Next, we project point B onto the plane E (in purple), which is a plane that passes through A and is parallel to the xy plane. Denote by P the projection of point B in E. (In other words, the* *vector AP is perpendicular to the normal vector of the plane E.) (a) Find the locus for the point P. Find the point B that will yield the maximum area for the triangle APB.*

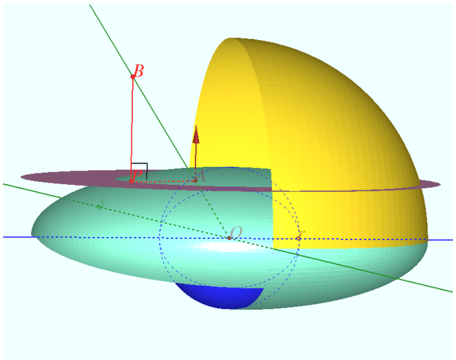

Figure 12. Generating an ellipsoid from two concentric spheres.

We write $A = (A_x, A_y, A_z), B = (B_x, B_y, B_z)$, and let $P = (P_x, P_y, P_z)$ be the locus point. We introduce the spherical coordinate system by letting φ the angle between OB and the positive $z-axis$, and the angle θ to be the angle between the projection of OB onto the xy-plane and the positive $x - axis$. If we let $a = OA$ and $b = OB$, then we see that $B_z = b\cos\varphi$, $B_x = b\sin\varphi\cos\theta$ and $B_y = b\sin\varphi\sin\theta$. We note that $P_z = A_z = a\cos\varphi$, $P_x = B_x = b\sin\varphi\cos\theta$ and $P_y = B_y = b\sin\varphi\sin\theta$. It shows that the locus surface in this case is an ellipsoid of the form

$$\frac{P_x^2}{b^2} + \frac{P_y^2}{b^2} + \frac{P_z^2}{a^2} = 1.$$

We may interpret part (a), finding the locus surface, as one way of constructing an ellipsoid as stated in *[5]*. We construct the locus surface in green as seen in Figure 12 with the help of *[4]*. We leave it as an exercise for the reader to find the maximum area for the triangle ABP.

3.4 Obtaining the Parametric Equation for an Ellipsoid from Spheres

Through exploring Example 8, we notice that it is not possible to construct an ellipsoid of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ if $a \neq b \neq c$ by using only two spheres. Thus, an important and similar question arising in this context is the following: How an ellipsoid in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, if $a \neq b \neq c$, can be expressed as $[a\cos\theta\sin\varphi, b\sin\theta\sin\varphi, c\cos\varphi]$,with $\theta \in [0, 2\pi]$ and $\varphi \in [0, \pi]$?

In order to answer this question, let us consider the following Figure 13, below.

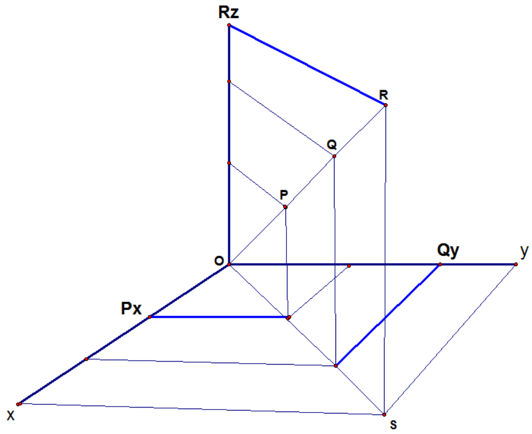

Figure 13 Spherical coordinate and projections.

Here, as mentioned before, we denote by φ the angle between $\overrightarrow{OR}$ and the $z-axis$ (pointing up), and by θ the angle $\overrightarrow{OS}$ and the $x-axis$. We pick three points, P, Q and R on three respective spheres of radii a, b and c such that P, Q and R are collinear. We project P onto the $xy-plane$ and obtain its $x-coordinate$, $P_x = a\sin\varphi\cos\theta$. Similarly, we obtain the $y-coordinate$ of Q, $Q_y = b\sin\varphi\sin\theta$. Finally, we use R and obtain its $z-coordinate$, $R_z = c\cos\varphi$. The locus surface of $[P_x, Q_y, R_z]$ will be the desired ellipsoid of the form $[a\cos\theta\sin\varphi, b\sin\theta\sin\varphi, c\cos\varphi]$,where $\theta \in [0, 2\pi]$ and $\varphi \in [0, \pi]$.

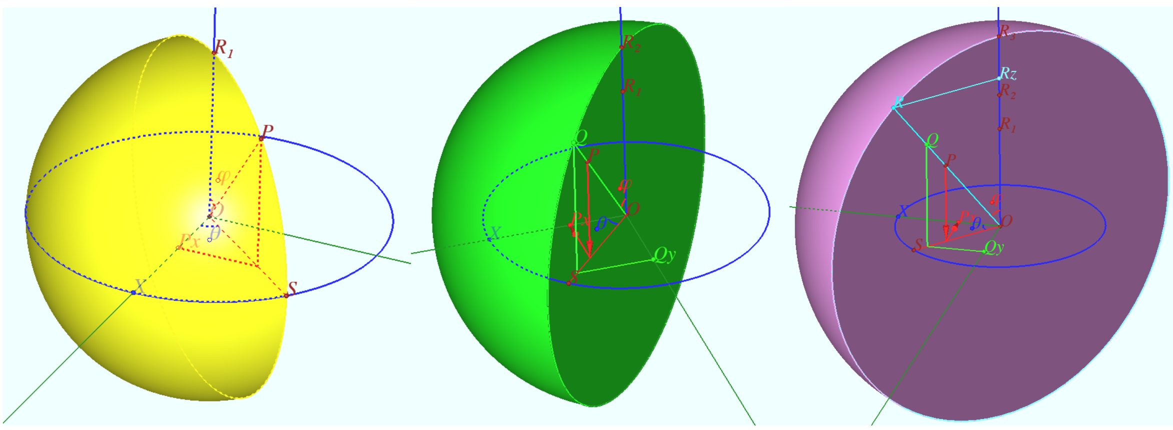

Figure 14(a). The point P and its x projection from the sphere of radius a.

Figure 14(b). The point Q and its y projection from the sphere of radius b.

Figure 14(c). The point R and its z projection from the sphere of radius c.

The expected locus ellipsoidal surface can be constructed with GInMA and we obtain the picture in Figure 15:

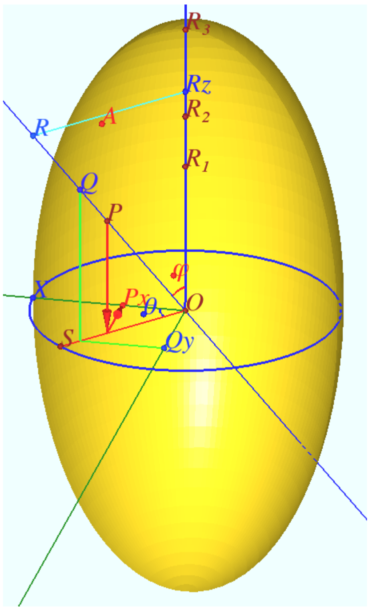

Figure 15. Final ellipsoidal locus surface.

3.5 Constructing Hyper-Ellipsoid in Higher Dimensions

In the same manner as we constructed a 3D ellipsoid using appropriate projections, we can analogously construct a 4D ellipsoid by setting

$$
\begin{aligned}
x_1 &= r_1 \cos\theta_1, \\
x_2 &= r_2 \sin\theta_1 \cos\theta_2 \\
x_3 &= r_3 \sin\theta_1 \sin\theta_2 \sin\theta_3, \text{ and} \\
x_4 &= r_4 \sin\theta_1 \sin\theta_2 \cos\theta_3.
\end{aligned}
$$

analogously with the help of four spheres of radii r_1, r_2, r_3 and r_4 respectively. In the similar manner, we may construct an $n-$dimensional ellipsoid geometrically of the following form through perpendicular projections of of $n-$ hyper spheres of radii $r_1, r_2, r_3, ..., r_n$, respectively:

$$
\begin{aligned}
x_1 &= r_1 \cos\theta_1, \\
x_2 &= r_2 \sin\theta_1 \cos\theta_2 \\
x_3 &= r_3 \sin\theta_1 \sin\theta_2 \sin\theta_3, \\
&\dots \\
x_{n-1} &= r_{n-1} \sin\theta_1 \sin\theta_2 ... \sin\theta_{n-1}, \text{ and} \\
x_n &= r_n \sin\theta_1 \sin\theta_2 ... \sin\theta_{n-2} \cos\theta_{n-1}.
\end{aligned}
$$

3.6 Exploring Other 3D Perpendicular Projections

Having arrived this far, we think that it is quite natural for students to make questions themselves about other scenarios, for example, replacing the outer sphere in Example 8 by an ellipsoid and try to see how the locus surface may change. In particular, we consider the following:

Exercise 9 *We are given a sphere of radius of* r_0, *centered at* $O = (0, 0, 0)$, *and an ellipsoid that is also centered at* $(0, 0, 0)$, *enclosing the given sphere. We are given the moving point* A *on the sphere and extend the ray* OA *to intersect the outer ellipsoid at a point* B. *Next, we project point* B *onto the plane* E, *which is a plane that passes through* A *and is parallel to the* xy *plane. Denote by* P *the projection of point* B *in* E. *In other words, the vector* AP *is perpendicular to the normal vector of the plane* E. *(a) Find the locus for the point* P *(b) Find the point* B *which yields the maximum area for the triangle* APB.

We write the parametric equation for the ellipsoid as $[a\cos\theta\sin\varphi, b\sin\theta\sin\varphi, c\cos\varphi]$, where φ denotes the angle between $\overrightarrow{AB}$ and positive $z-axis$ and θ denotes the angle between the projection of OB onto the xy-plane and the positive $x-axis$. Then the locus $[P_x, P_y, P_z]$ can be written as follows:

$$\begin{aligned} P_x &= B_x = x(\theta, \varphi), \\ P_y &= B_y = y(\theta, \varphi), \\ P_z &= A_z = r_0 \cos\varphi. \end{aligned}$$

We leave it as an exercise for the reader to find the maximum area for the triangle ABP.

In order to generalize the idea of obtaining a locus through perpendicular projections, we replace the outer ellipsoid in Exercise 9 with another surface that encloses a sphere of a given radius. Specifically, we consider the following cardioid surface below:

Example 10 *We are given a sphere centered at* $O = (0, 0)$ *with radius of* r_0, *and the cardioid surface* S, *by rotating* $[x(t), y(t)] = [a(1-\cos t)\cos t + a, a(1-\cos t)\sin t]$, *where* $t \in [0, 2\pi]$, *around the* $x-axis$. *Let* A *be a moving point on the sphere and we extend the ray* OA *to intersect the outer cardioid surface at a point* B. *Next, we project point* B *onto the plane* E, *which is a plane that passes through* A *and is parallel to the* xy *plane. Denote by* P *the projection of point* B *in* E. *In other words, the vector* AP *is perpendicular to the normal vector of the plane* E. *(a) Find the locus for the point* P *(b) Find the point* B *which yields the maximum area for the triangle* APB.

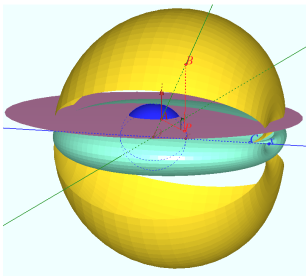
Figure 16(a) A sphere, cardioidal surface and locus

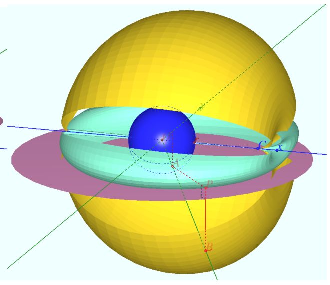
Figure 16(b) Locus surface when the point A varies

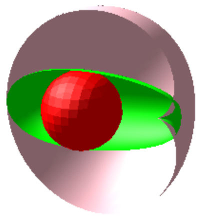
Figure 16(c) Locus generated by MAPLE

As mentioned in Exercise 4, the cardioid surface can be written as $[x(t), y(t)\cos\varphi, y(t)\sin\varphi]$,where $t \in [0, 2\pi]$ and $\varphi \in [0, \pi]$. As we have seen in Example 7 the locus for $[x(t), y(t)]$ is $[x^*(t), y^*(t)] = \left[a\left(\sin^2 t + \cos t\right), r_0\left(\frac{\sin t(1-\cos t)}{\sqrt{2-\cos^2 t}}\right)\right]$. Thanks to symmetry, the locus surface for the cardioid surface is $[x^*(t), y^*(t)\cos\varphi, y^*(t)\sin\varphi]$. In Figures (16)(a) and (b), with the aid of GInMA *[4]*, we plotted various views of cardioid surfaces together with the enclosed spheres and respective locus surfaces. We also verified the locus surface analytically with [6] when $a = 2$ and $r_0 = 1$ as displayed in Figure 16(c).

3.7 Possible Real Life Interpretation in 3D

1. A sea rock is similar to the shape of half of a small sphere. An airplane is flying on a path of C, which lies on the surface of an ellipsoid or a cardioid surface. We assume the ellipsoidal or cardioid surface is enclosing the sphere. The airplane decides to lower a basket which tied to a vertical ladder to rescue people who are stuck in the sea rock. We assume those people who need to be rescued from the sea rock may need to swim to the location where the basket is lowered. But because of the tides, sea rock will be covered by various levels of waters at times. The tides are planes that pass through a moving point A on the sea rock and are parallel to the sea level. (a) Find the locus of the rescuing basket. (b) Furthermore, if it is decided that the best place the airplane should drop the basket is at the point when the area of the triangle ABP reaches its maximum. Find the place where the airplane should drop the basket.

2. Exercise for the reader: Use your imagination to interpret one of your real-life scenarios.

4 Discussions

Following the ideas of how we may construct an ellipsoid geometrically through its parametric equation, we can also construct any known 3D parametric equations from three respective surfaces geometrically through perpendicular projections. Obviously, such construction is not unique. In general, the radii of the three respective spheres depends on the (u, v)-coordinates.

In the case of the ellipsoid $[x(u,v), y(u,v), z(u,v)] = [a\sin v\cos u, b\sin v\sin u, c\cos v]$, the radii for three respective spheres are constant, namely, $r_1 = a, r_2 = b$, and $r_3 = c$. Furthermore, when

$$\frac{x(u,v)}{\cos u} = \frac{y(u,v)}{\sin u}, \frac{x(u,v)}{\sin v\cos u} = \frac{z(u,v)}{\cos v}, \text{ or } \frac{y(u,v)}{\sin v\sin u} = \frac{z(u,v)}{\cos v},$$

then two of the spheres would have the same radius as we have seen from Example 8. We use the following example as a demonstration.

$$\begin{aligned} x(u,v) &= a^2(\sin(v)\sin(2u)/2), && (7)\\ y(u,v) &= a^2(\sin(2v)\cos^2(u)), && (8)\\ z(u,v) &= a^2(\cos(2v)\cos^2(u)). && (9)\end{aligned}$$

When $a = 3$ we can construct this surface with Maple *[6]*, as illustrated in Figure 17 below.

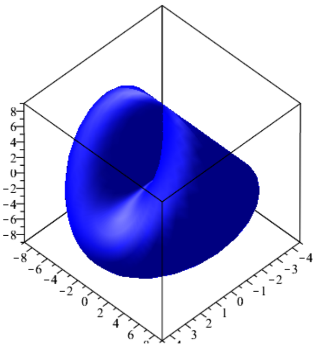

Figure 17. A cross-cap surface parameterized by Eqs (7)-(9) when $a = 3$.

It is easy to see the cross-cap surface $[x(u,v), y(u,v), z(u,v)]$ can be the locus surface of three respective closed surfaces through perpendicular projections.

The next example is interesting because it shows that we may construct a cross-cap surface from two spheres by choosing the respective radii appropriately. The next example shows that we may construct a type of cross-cap surface from two spheres by properly choosing their respective radii appropriately. We illustrate this in the following

Example 11 *Let spheres S_1 and S_2 be centered at the origin and with radii of $a\cos v$ and $\frac{a(\cos v - \cos^2 u\sin v\tan v)}{2}$ respectively. Let P be an arbitrary point on S_1 and make its projection onto $x-axis$ to obtain its x-coordinate $P_x = a\cos v\sin v\cos u$, and also onto $y-axis$ to get its y-coordinate $P_y = a\cos v\sin v\sin u$. Next, let Q be an arbitrary point on S_2 and project it onto its $z-$axis to obtain its $z-$coordinate of $Q_z = \left(\frac{a(\cos v - \cos^2 u\sin v\tan v)}{2}\right)\cos v = \frac{a(\cos^2 v - \sin^2 v\cos^2 u)}{2}$. Now, the parametric equation of $[P_x, P_y, Q_z]$ represents a cross-cap locus surface construction* *with the aid of GInMA [4], which can be seen in Figures 18(c).*

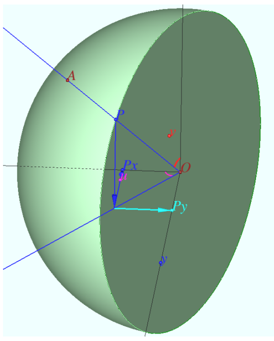

Figure 18(a). Project a point P onto x and y coordinates.

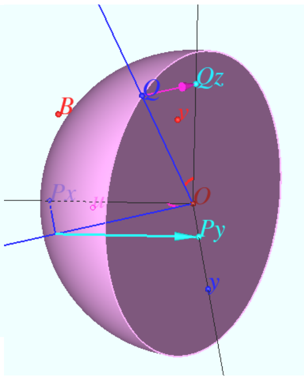

Figure 18(b). Project a point Q onto the z coordinate.

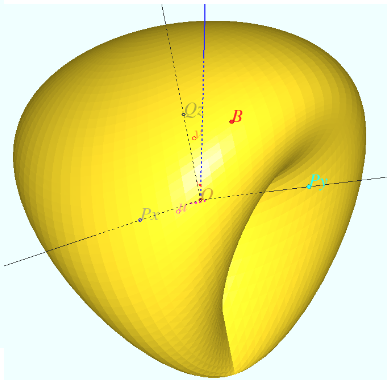

Figure 18(c). The resulting cross-cap locus surface.

5 Conclusions

Examinations alone should not be the sole measurement of a student's success. It should be as important to see how a math curriculum includes proper components of exploration with the help of technological tools, especially where real life applications can be found. In an article (see *[9]*), it is stated that 'Taiwan plans a radical reform of its education system, one aiming to set it apart in East Asia by playing up creativity and student initiative instead of the rote memorization that dominates classroom learning in this part of the world.' While many educators, researchers and parents would applaud this brave and bold initiative, how a government would implement this agenda remains to be seen. It is not how to say the right thing but how to develop strategies to see it through.

Therefore, we outline some of the necessary knowledge a teacher must be familiar with, so technological tools can be integrated in a math curriculum in order to motivate more students to be interested in the the area of STEM (Science, Technology, Engineering and Mathematics) area.

1. Use a DGS to simulate animations in two dimensions.
2. Encourage students to make conjectures through observations made in step 1.
3. Encourage students to verify their results using a CAS for 2D case.
4. Extend students' observations to a 3D scenarios wherever possible.
5. Prove corresponding results for 3D cases analytically using a CAS if possible.
6. Extend results to finite dimensions or beyond wherever possible.

For example, in this paper we turned two static college entrance exam practice problems into interesting exploratory scenarios, both in 2D and 3D settings. We notice that the required mathematical knowledge of those extended 3D problems are accessible to high school students once they are familiar with parametric equations of 3D surfaces. Nevertheless, we remark the necessity of developing more 3D DGS for visualizing purposes. Allowing users to drag and view dynamic figures from different perspectives is clearly beneficial and assists them before they attempt to set up possibly complex algebraic equations.

It is common sense that teaching to a test can never promote creative thinking skills, it could even lose potential students who might pursue mathematics related fields in the future. We know that addressing the importance and timely adoption of technological tools in teaching, learning and research can never be wrong. Finally, we should consider selecting those examples that can be explored from middle to high schools, university levels, or even beyond when learners have acquired the necessary content knowledge. Similarly, we need applications that are STEM related and link mathematics to real-world applications wherever possible. Access to technological tools has motivated us to rethink how mathematics can and should be presented more interestingly and also how mathematics can be made a more cross disciplinary subject. There is no doubt that evolving technological tools have helped learners to discover mathematics and to become aware of its applications.

6 Acknowledgements

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7 Electronic Supplementary Materials

[S1] GInMA file for Example 1 and Exercise 2:
https://mathandtech.org/eJMT_June_2017/eJMT_Example1_Exercise2.ginma.

[S2] Maple file for Example 1 and Exercise 2:
https://mathandtech.org/eJMT_June_2017/eJMT_Example1_Exercise2.mws.

[S3] GInMA file for Example 3 and Exercise 4:
https://mathandtech.org/eJMT_June_2017/eJMT_Example3.ginma.

[S4] Maple file for Example 3 and Exercise 4:
https://mathandtech.org/eJMT_June_2017/eJMT_Example3_Exercise4.mws.

[S5] GInMA file for Example 6 and Example 7:
https://mathandtech.org/eJMT_June_2017/eJMT_Example6_Example7.ginma.

[S6] Geometry Expressions file for Example 5:
https://mathandtech.org/eJMT_June_2017/eJMT_Example5.gx.

[S7] Geometry Expressions file for Example 7:
https://mathandtech.org/eJMT_June_2017/eJMT_Example7.gx.

[S8] GInMA file for Example 8:
https://mathandtech.org/eJMT_June_2017/eJMT_Example8.ginma.

[S9] GInMA file for Section 3.4:
https://mathandtech.org/eJMT_June_2017/eJMT_Section3.4.ginma.

[S10] GInMA file for Example 10:
https://mathandtech.org/eJMT_June_2017/eJMT_Example10.ginma.

[S11] Maple file for Example 10:
https://mathandtech.org/eJMT_June_2017/eJMT_Example10.mws.

[S12] GInMA file for Example 11:
https://mathandtech.org/eJMT_June_2017/eJMT_Example11.ginma.

[S13] A video clip summarizing those GInMA examples:
https://mathandtech.org/eJMT_June_2017/eJMT_June2017.mp4.

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[4] Geometry in Mathematical Arts (GInMA): *A Dynamic Geometry System*, see http://deoma-cmd.ru/en/Products/Geometry/GInMA.aspx. Free version can be downloaded to experiment GInMA files.

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